Chapter Six.

Solving equations.

The *Preliminary Work* section at the beginning of this book reminded us of work encountered previously involving

> *using a formula to determine the value of a variable, or pronumeral, that appears in the formula by itself and on one side of the equals sign, given the values of the variables, or pronumerals appearing on the other side.*

In the examples given there we used $A = P + I$ to determine A, given P and I,

$C = 2\pi r$ to determine C knowing π and given r,

$s = ut + \frac{1}{2}at^2$ to determine s, given u, a and t.

Suppose instead, in this last formula, we were asked to determine u given s, t and a.

Substituting the given values into the formula gives us a statement of equality, called an **equation**, which will have to be **solved** to determine the unknown quantity.

For example, if $s = 63$, $t = 3$ and $a = 10$, $s = ut + \frac{1}{2}at^2$ becomes

$$63 = u(3) + \frac{1}{2}(10)(3)^2$$

i.e.

$$63 = 3u + 45$$

Solving this **equation** means finding the value of u for which $3u + 45$ does equal 63.

What is the required value of u for which

$$3u + 45 = 63 \ ?$$

Note: The word *variable* was used in chapter one when considering data. In that case there were *various* responses that could be given to a question like "What is your favourite colour?" Hence the use of the word variable there.

In the formula $A = P + I$, A can take *various* values, dependent upon the values of P and I. Hence the use of the word variable in this situation. Given specific values of P and I, the particular value of A can be determined.

Solving equations.

To solve $3u + 45 = 63$ did you proceed mentally,

or did you perhaps use the ability of some calculators to solve equations

or did you carry out a step by step process to isolate u ?

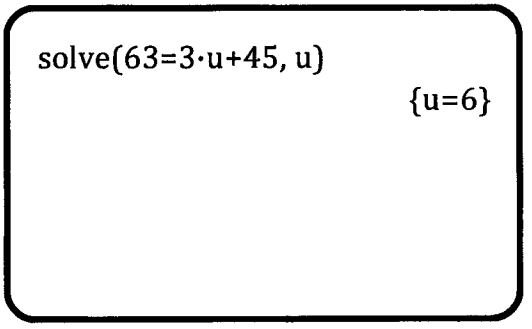
These three approaches are shown below and in the examples that follow.

- Solving $3u + 45 = 63$ mentally:

  We know that *eighteen* plus forty five equals 63.
  Thus $3u = 18$.
  But three multiplied by *six* is eighteen.
  Hence $u = 6$.

Or ..

- Using the solve facility on a calculator.



Or ..

- Using a step by step process of doing something to both sides of the equation in order to isolate the unknown whilst retaining the correctness of the equality statement.

Starting with the given equation:	$63 = 3u + 45$
We subtract 45 from each side to isolate $3u$:	$18 = 3u$
We divide each side by 3 to isolate u:	$6 = u$
Thus	$u = 6$

Example 1

Solve the following equations

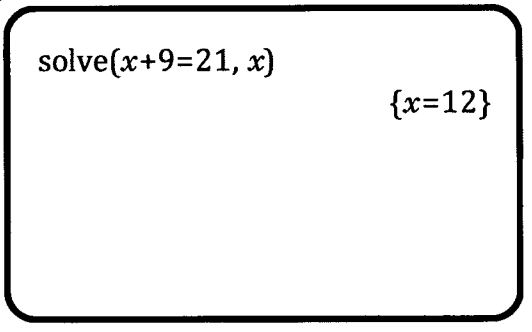
(a) $x + 9 = 21$, (b) $5x - 7 = 23$, (c) $15 - 2x = 4$,

(a) $x + 9 = 21$

Mentally:

We know that *twelve* add nine equals twenty one. Thus $x = 12$.

Using the solve facility:



Step by step approach to isolate x:

We are given the equation: $x + 9 = 21$

We subtract 9 from each side to isolate x: $x = 21 - 9$

$= 12$

Thus $x = 12$

(b) $5x - 7 = 23$

Mentally:

We know that *thirty* take seven equals twenty three.

Thus $5x = 30$.

But five times *six* equals 30 and so $x = 6$.

Using the solve facility:

solve(5x−7=23, x)

{x=6}

Step by step approach to isolate x:

We are given the equation: $5x - 7 = 23$

We add 7 to both sides to isolate $5x$: $5x = 23 + 7$

$= 30$

Now we divide both sides by 5 to isolate x: $x = 30 \div 5$

$= 6$

Thus $x = 6$

(c) $15 - 2x = 4$

Mentally:

We know that fifteen take *eleven* equals four.

Thus $2x = 11$ and so $x = 5{\cdot}5$.

Using the solve facility:

solve(15−2x=4, x)

{x=5.5}

Step by step approach to isolate x:

We are given the equation: $15 - 2x = 4$

We add $2x$ to both sides to make the x term positive: $15 = 4 + 2x$

We subtract 4 from both sides to isolate $2x$: $15 - 4 = 2x$

$\therefore \quad 11 = 2x$

Now we divide both sides by 2 to isolate x: $5{\cdot}5 = x$

$\therefore \quad x = 5{\cdot}5$

Equations with brackets or fractions.

Some equations may involve brackets. For example $3(2x + 1) - 5 = 40$.

Some equations may involve fractions. For example $\frac{2x + 3}{5} = 4$.

Example 2

Solve the following equations (a) $3(x - 1) = 21$,

(b) $3(2x + 1) - 5 = 40$,

(c) $\frac{2x + 3}{5} = 4$.

(a) $3(x - 1) = 21$

Mentally:

We know that three times *seven* equals twenty one.

Thus $x - 1 = 7$.

But *eight* take one equals 7 and so $x = 8$.

Using the solve facility:

solve(3(x−1)=21, x)

{x=8}

Step by step approach to isolate x:

We are given the equation:	$3(x - 1) = 21$
Expand to remove bracket:	$3x - 3 = 21$
Add 3 to both sides to isolate $3x$:	$3x = 24$
Now divide both sides by 3 to isolate x:	$x = 8$

(b) $3(2x + 1) - 5 = 40$

Mentally:

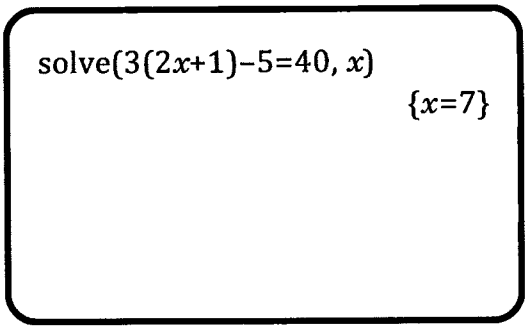
We know that three *fifteens* take away five is equal to 40.

Thus $2x + 1 = 15$.

But *fourteen* add one is equal to fifteen.

Hence $2x = 14$ and so $x = 7$.

Using the solve facility:



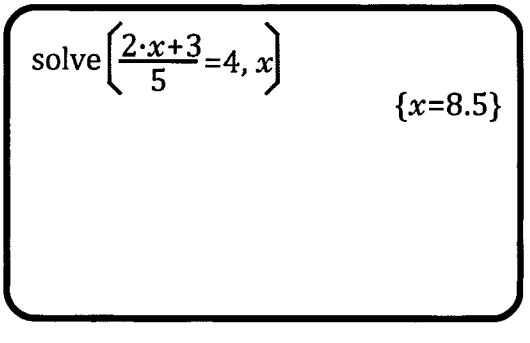
Step by step approach to isolate x:

We are given the equation:	$3(2x + 1) - 5 = 40$
Expand to remove bracket:	$6x + 3 - 5 = 40$
Hence	$6x - 2 = 40$
Add 2 to both sides to isolate $6x$:	$6x = 42$
Now divide both sides by 6 to isolate x:	$x = 7$

(c) $\dfrac{2x + 3}{5} = 4$

Mentally:
We know that *twenty* divided by five is equal to four.
Thus $2x + 3$ must equal 20.
But *seventeen* plus three is equal to twenty.
Hence $2x = 17$ and so $x = 8{\cdot}5$.

Using the solve facility:



Step by step approach to isolate x:

We are given the equation:	$\dfrac{2x + 3}{5} = 4$
Multiply both sides by 5 to remove fractions:	$2x + 3 = 20$
Subtract 3 from both sides to isolate $2x$:	$2x = 17$
Now we divide both sides by 2 to isolate x:	$x = 8{\cdot}5$

If the equations are more involved the mental approach can be too difficult but we can still use either
- the solve facility on a graphic calculator,

or
- the step by step process to isolate the unknown,

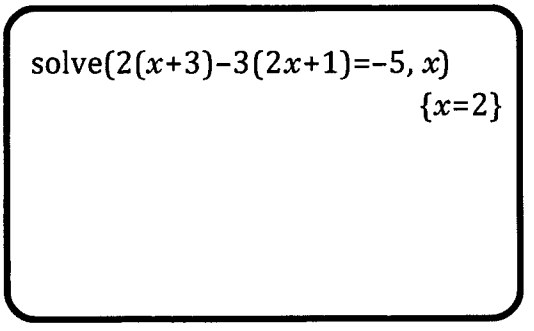
as shown below.

Example 3

Solve the following equations: (a) $2(x+3)-3(2x+1)=-5$

(b) $\frac{x}{7}=\frac{3}{10}+\frac{2x}{21}$

(a) **Using the solve facility:**



Step by step approach to isolate x:

We are given the equation: $2(x+3)-3(2x+1) = -5$

Expand to remove brackets: $2x+6-6x-3 = -5$

Collect like terms: $-4x+3 = -5$

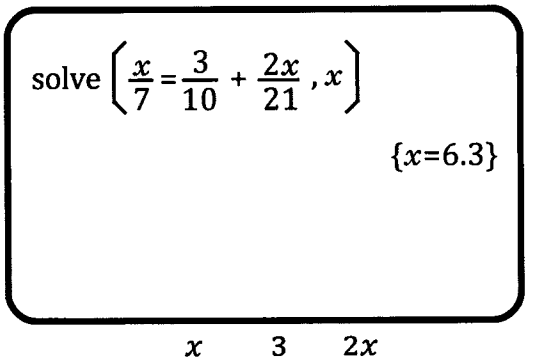
We add $4x$ to both sides to make the x term positive: $3 = -5+4x$

Add 5 to both sides to isolate $4x$: $8 = 4x$

Now we divide both sides by 4 to isolate x: $2 = x$

$\therefore \quad x = 2$

(b) **Using the solve facility:**



Step by step approach to isolate x:

We are given the equation: $\frac{x}{7} = \frac{3}{10}+\frac{2x}{21}$

Multiply both sides by 210 to remove fractions: $210\times\frac{x}{7} = 210\times\frac{3}{10}+210\times\frac{2x}{21}$

i.e.: $30x = 63+20x$

Subtract $20x$ from each side: $10x = 63$

Now we divide both sides by 10 to isolate x: $x = 6{\cdot}3$

Example 4

A firm manufacturing a particular motorbike determines that the profit, $\$P$, made from the production and sale of x of these bikes is given by

$$P = 5\,400x - 238\,000$$

Calculate the number of these bikes the firm must produce and sell to make a profit that exceeds one million dollars.

Either

Use the ability of some calculators to determine the value of x when $P = 1\,000\,000$ in the given formula.

Equation:
P=5400·x−238000
○ P=1000000
◉ x=229.259259259259
Lower=−9E+999
Upper=9E+999

Or

Substitute $P = 1\,000\,000$ into the given formula and use the step by step approach to determine x, as shown below.

We are given the formula: $P = 5\,400x - 238\,000$

Substitute $P = 1\,000\,000$: $1\,000\,000 = 5\,400x - 238\,000$

Add 238 000 to both sides to isolate $5\,400x$: $1\,238\,000 = 5\,400x$

$\therefore \quad x \approx 229{\cdot}3$

However the situation requires x to take positive integer values and so:
The firm must produce and sell at least 230 of these bikes to make a profit that exceeds one million dollars.

Example 5

Formula: $\boxed{s = \frac{(u + v)}{2} t}$. Find t given that $s = 35$, $u = 10$ and $v = 4$.

By calculator.

Equation:
$s = \frac{u+v}{2} \cdot t$
○ s=35
○ u=10
○ v=4
◉ t=5
Lower=−9E+999
Upper=9E+999

Substitute and then isolate t.

$$s = \frac{(u + v)}{2} t$$

$$35 = \frac{(10 + 4)}{2} t$$

$$\therefore \quad 35 = 7t$$

$$\therefore \quad t = 5$$

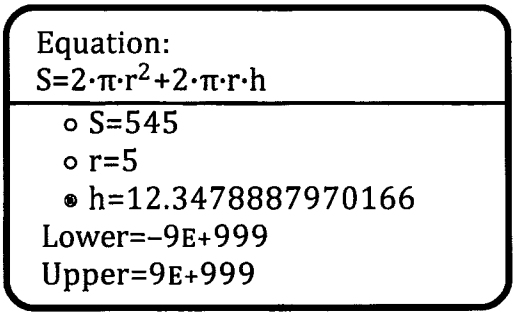
Example 6

Formula: $\boxed{S = 2\pi r^2 + 2\pi rh}$

Find h given that $S = 545$ and $r = 5$ giving your answer correct to 1 decimal place.

Either

Determine the value of h for the given values of S and r using the ability of some calculators to determine unknown values in a formula given sufficient information.



Or

Substitute $S = 545$ and $r = 5$ into the given formula and use the step by step approach to determine h, as shown below.

We are given the formula: $S = 2\pi r^2 + 2\pi rh$

Substitute $S = 545$ and $r = 5$: $545 = 2\pi(5)^2 + 2\pi(5)h$

$\therefore \quad 545 = 50\pi + 10\pi h$

Subtract 50π from both sides to isolate $10\pi h$: $545 - 50\pi = 10\pi h$

Now we divide both sides by 10π to isolate h: $\dfrac{545 - 50\pi}{10\pi} = h$

$\therefore \quad h = 12{\cdot}3$ (to one d.p.)

Thus when $S = 545$ and $r = 5$, $h = 12{\cdot}3$ (to one d.p.).

Linear equations.

The six equations shown below are all examples of **linear equations in one variable**.

$2x + 17 = 5$ $\qquad$ $5p - 7 = 32$ $\qquad$ $15 - 3z = 6$

$\dfrac{w}{2} = 5$ $\qquad$ $2(3q - 5) + 1 = 15$ $\qquad$ $\dfrac{2n+3}{5} = 4$

Each equation, after expansion of any brackets and separation of fractions, only involves terms that are either just a number, or the variable multiplied or divided by a number. Linear equations do <u>not</u> involve the variable squared (x^2), cubed (x^3), square rooted ($\sqrt{x}$), in the denominator of a fraction $\left(\dfrac{3}{x-1}\right)$, as a power ($2^x$) etc.

Each of the above equations can, with a bit of work, be written in the form $ax + b = 0$, the basic form of a linear equation in one variable:

$2x + 12 = 0$ $\qquad$ $5p - 39 = 0$ $\qquad$ $-3z + 9 = 0$

$w - 10 = 0$ $\qquad$ $6q - 24 = 0$ $\qquad$ $2n - 17 = 0$

(As we will see later in this text, and as you may already be familiar with, equations of the form $y = ax + b$ give straight line, or **linear**, graphs.)

The emphasis in this text will be on solving **linear** equations.

Exercise 6A

(Use this exercise to practise the various methods shown in the previous pages.)

1. Solve the following equations.

(a) $x + 5 = 11$ (b) $5 - x = 31$ (c) $x + 3 = 31$

(d) $3x + 7 = 25$ (e) $15 - 2x = 6$ (f) $3x - 7 = 2$

(g) $2(x + 3) = 14$ (h) $3(x - 1) = 21$ (i) $5(x + 2) = 15$

(j) $2(x - 5) = 16$ (k) $3(1 + x) = 18$ (l) $5(2x - 1) = 9$

(m) $\frac{x}{3} = 5$ (n) $\frac{x}{2} = 21$ (o) $\frac{3x}{10} = 1{\cdot}5$

(p) $\frac{5x}{7} = 1$ (q) $\frac{x}{7} = 12$ (r) $\frac{3x - 5}{2} = 8$

(s) $3(x + 2) + 5(2x - 1) = 27$ (t) $7(2x + 3) - 3(2x + 1) = 10$

(u) $\frac{x}{2} + 5 = 11$ (v) $\frac{x}{7} = \frac{6}{21}$ (w) $\frac{4x}{3} + \frac{3}{4} = -\frac{x}{6}$

(x) $\frac{x}{2} - \frac{2x - 1}{5} = 2$ (y) $\frac{3x + 1}{5} + \frac{5x - 1}{4} = 24$

2. $\boxed{A = P + I}$

(a) Find P given that $A = 676$ and $I = 26$.
(b) Find A given that $P = 1250$ and $I = 85$.
(c) Find I given that $P = 1185$ and $A = 1240$.

3. $\boxed{v = u + at}$

(a) Find v given that $u = 14$, $a = 2$ and $t = 3$.
(b) Find u given that $v = 30$, $a = 3$ and $t = 4$.
(c) Find a given that $v = 30$, $u = 16$ and $t = 7$.
(d) Find a given that $v = 8$, $u = 20$ and $t = 4$.
(e) Find t given that $v = 23$, $u = 5$ and $a = 6$.
(f) Find t given that $v = 12$, $u = -10$ and $a = 2$.

4. $\boxed{C = 2\pi r}$

(a) Find r given that $C = 25$. (Answer correct to two decimal places.)
(b) Find r given that $C = 95$. (Answer correct to two decimal places.)
(c) Find C given that $r = 8$. (Answer correct to two decimal places.)
(d) Find r given that $C = 128\pi$.

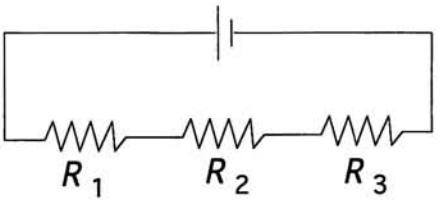
5. $\boxed{A = 2\pi rh}$

(a) Find A given that $r = 1$ and $h = 4$. (Answer correct to 2 d.p.)
(b) Find r given that $A = 125$ and $h = 7$. (Answer correct to 2 d.p.)
(c) Find h given that $A = 200$ and $r = 6$. (Answer correct to 2 d.p.)

6. $\boxed{s = \frac{(u + v)}{2} t}$

(a) Find t given that $s = 56$, $u = 3$ and $v = 5$.
(b) Find u given that $s = 92$, $v = 14$ and $t = 8$.
(c) Find v given that $s = 22{\cdot}5$, $u = -6$ and $t = 5$.

7. If three resistors, R_1, R_2 and R_3 are placed in an electrical circuit as shown in the diagram they are said to be in **series**. The total resistance, R, is then given by:



$$R = R_1 + R_2 + R_3$$

(a) Find R if $R_1 = 2$, $R_2 = 5$, $R_3 = 4$.
(b) Find R if $R_1 = 10$, $R_2 = 15$, $R_3 = 20$.
(c) Find R_1 if $R = 32$, $R_2 = 8$, $R_3 = 14$.
(d) Find R_3 if $R = 80$, $R_1 = 25$, $R_2 = 28$.

8. $\boxed{S = 2\pi r^2 + 2\pi rh}$

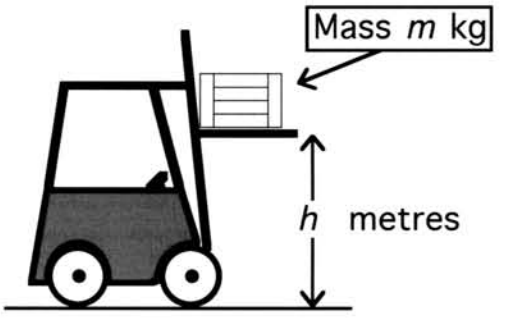
(a) Find h given $S = 250$ and $r = 4$. (Answer correct to two decimal places.)
(b) Find h given $S = 1000$ and $r = 4$. (Answer correct to two decimal places.)

9. A certain mass of gas is held under pressure at a constant temperature. V, the volume of the gas, is related to the pressure according to the rule:

$$V \times P = 150.$$

Find (a) V given that $P = 6$, (b) P given that $V = 15$,
(c) V given that $P = 7{\cdot}5$, (d) P given that $V = 60$.

10. When a fork lift truck lifts an object of mass m kg through a height h metres it gives the object potential energy equal to P Joules where $P = 9{\cdot}8\,mh$.



(a) Find P when $m = 50$ and $h = 1{\cdot}5$.
(b) Find P when $m = 400$ and $h = 2$.
(c) Find m when $P = 2\,940$ and $h = 2$.
(d) Find h when $P = 13\,230$ and $m = 600$.

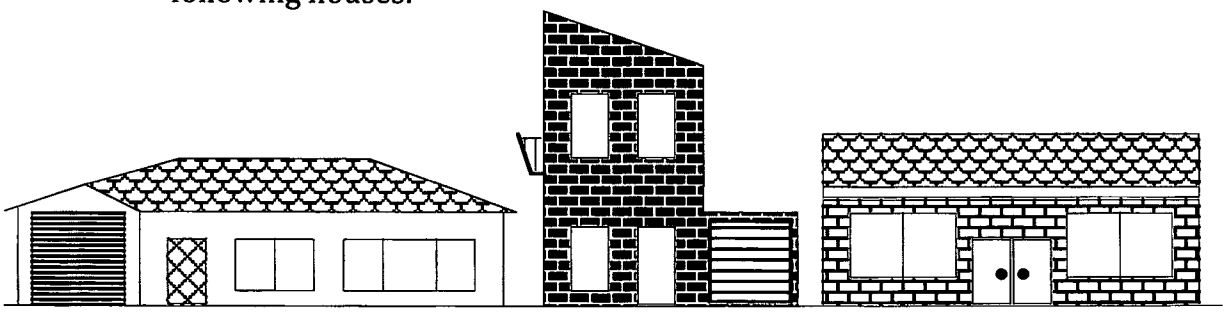
11. A car rental firm charges \$$C$ for renting a particular vehicle where C depends on d, the number of days hired, and k, the number of kilometres travelled. C is calculated according to the rule: $C = 30 + 70d + 0{\cdot}1k$.

(a) Find the cost of renting the vehicle for 7 days and travelling 500 km.
(b) Find the cost of renting the vehicle for 3 days and travelling 1200 km.
(c) A person wishes to rent the vehicle for six days but does not want the hire costs to exceed \$700. How far could the person travel in the car in the six days?

12. When an agent of a particular real estate company sells a house for A thousand dollars the commission the agent receives is $\$P$ where

$$P = 600 + 4A$$

(a) Find the amount the agent receives in commission for the sale of each of the following houses:



$625 000 $700 000 $375 000

(b) An agent sells a house and receives a commission of $2380. How much was the house sold for?

(c) An agent sells a house and receives a commission of $4000. How much was the house sold for?

13. If, in an archaeological dig, human bones are found, or if in a macabre murder case parts of a body are discovered, these bones can be used to estimate the height the person was when alive. In particular the bone in the upper arm from elbow to shoulder, called the humerus, is a good indicator of height.
If the humerus is of length h cm then a reasonable estimate for the height of a male with this length humerus is: $(2{\cdot}9h + 71)$ cm,
and a reasonable estimate for the height of a female with this length humerus is:

$$(2{\cdot}75h + 71) \text{ cm.}$$

(a) In an archaeological dig the remains of a male are uncovered and the humerus is found to be 34 cm long. Estimate the height of the male.

(b) What would be the expected humerus length of a 1·81 metre tall female?

14. A company has five thousand calendars printed. If it sells x of these calendars, where x is from a low of zero to a high of 5000, the profit produced will be $\$P$ where P is given by: $P = 12{\cdot}7x - 29\,750$

(a) What will be the profit if the firm sells 2 500 of the calendars?

(b) What will be the profit if the firm sells 3 500 of the calendars?

(c) What will be the profit if the firm sells all but 800 of the calendars?

(d) What is the least number of the calendars the firm needs to sell to make a profit that exceeds $9 000?

(e) If the firm are left with 3200 of the calendars unsold determine whether they have made a profit or a loss and state how much?

(f) What is the greatest profit the firm can make from this venture?

(g) If the firm sold none of the calendars how much would they lose?

(h) What is the least number of calendars the firm must sell to avoid making a loss?

Miscellaneous Exercise Six.
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. Solve the following equations.

   (a) $3x - 1 = 20$ (b) $5x + 6 = 24 - 4x$

   (c) $3x + 7 = 21 - x$ (d) $6(2x + 1) - 5 = 31$

   (e) $\frac{x-2}{3} - 3 = 1$ (f) $\frac{2x}{9} = \frac{5}{3}$

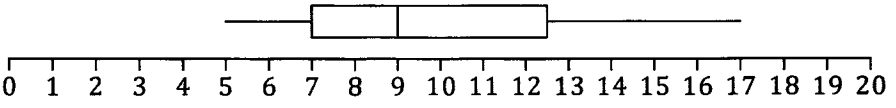
2. Formula: $\boxed{A = 4bh + b^2}$

   (a) Find A given that $b = 3$, $h = 5$. (b) Find h given that $A = 119$, $b = 7$.

3. The following nine scores are listed in ascending order, from left to right.

   a + 2, a + 3, b, c − 1, c − 1, c + 1, d, e − 1, a + e.

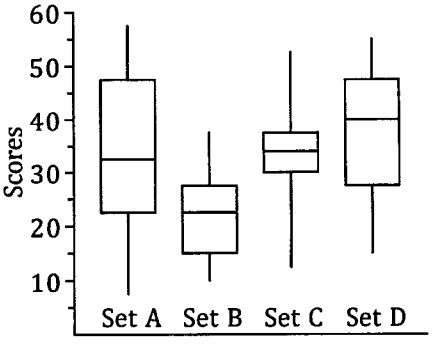
   The box and whisker diagram for these nine scores is as follows:



   Determine a, b, c, d and e.

4. To pass a particular course a student needs to gain a mean of at least 60% in the six tests that form the course assessment. In the first five tests the student achieves marks of 65%, 58%, 71%, 60% and 59%. What percentage mark does the student require in test six in order to pass the course?

5. The box plots on the right are for four sets of data, A, B, C and D. Which of the data sets ...
   (a) seems to involve the greatest variability?
   (b) has the smallest interquartile range?
   (c) has the smallest range?
   (d) contains the lowest of all the scores?
   (e) could the following apply to:
   More than half of the scores in set _ exceed all of the scores in set _.



6. The marks obtained by 47 students in an examination marked out of 140 are shown below:

25	32	50	54	59	66	67	69	71	73	75	75
76	76	78	80	81	82	83	83	84	85	88	88
89	89	89	89	90	92	94	95	96	99	100	104
106	108	109	109	111	111	113	114	115	115	117	

With the mean of these marks being $\bar{x}$ and the standard deviation σ, grades are awarded to these 47 students as follows

		exam mark	$\geq$	$\bar{x} + 1{\cdot}25\sigma$	Grade A
$\bar{x} + 0{\cdot}5\sigma$	$\leq$	exam mark	$<$	$\bar{x} + 1{\cdot}25\sigma$	Grade B
$\bar{x} - 0{\cdot}5\sigma$	$\leq$	exam mark	$<$	$\bar{x} + 0{\cdot}5\sigma$	Grade C
$\bar{x} - 2\sigma$	$\leq$	exam mark	$<$	$\bar{x} - 0{\cdot}5\sigma$	Grade D
		exam mark	$<$	$\bar{x} - 2\sigma$	Fail grade

Determine the number of students obtaining each grade.

7. A real estate agent in a particular region wants to publish the average price of houses sold in the town each month. Past figures indicate that each month somewhere between 10 and 50 houses are sold each month. Most houses in the area are of a similar nature except for a small number of beachside luxury properties. These do not come up for sale very often but when they do they are priced very much above most others in the region.
Which average, mean, median or mode, would you advise the agent to use for the average monthly price and why?

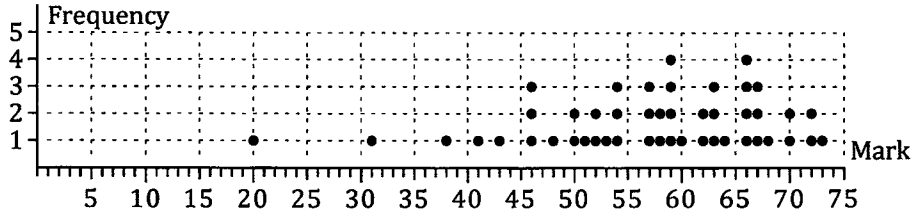
8. The salaries of the 187 full time employees of a large manufacturing company were distributed as follows:

Category	Salary \$$S$			Number of Employees.
A	40 000	$\leq S <$	50 000	23
B	50 000	$\leq S <$	60 000	64
C	60 000	$\leq S <$	70 000	43
D	70 000	$\leq S <$	80 000	25
E	80 000	$\leq S <$	90 000	14
F	90 000	$\leq S <$	100 000	9
G	100 000	$\leq S <$	110 000	5
H	110 000	$\leq S <$	120 000	3
I	120 000	$\leq S <$	130 000	1

(a) Calculate the mean and standard deviation of this distribution.

(b) With increased automation in the manufacturing processes the company no longer requires such a large workforce. Through voluntary redundancy and non-replacement of retirees the company reduced its workforce to 173 by losing 5 employees from category A, 4 from category B, 3 from category C and 2 from category F. Calculate the mean and standard deviation of the salaries of this workforce of 173.

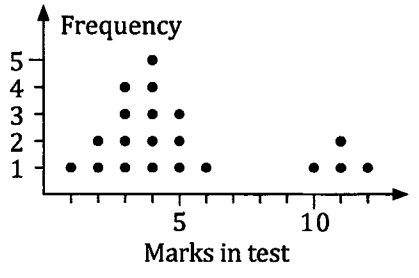
9. One hundred and twenty seven people applied for a particular job vacancy and the company involved decided to invite what they considered to be the best 50 applicants to take an aptitude test. On test day 47 of the invited 50 turned up and the marks obtained (out of 75) were as shown in the following diagram:



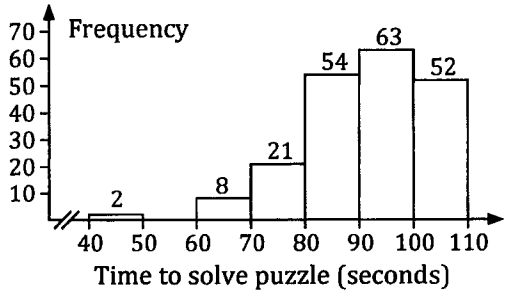
The company decides to invite for interview any of the applicants who achieved a mark in the test that is more than 1 standard deviation above the mean. How many of the applicants do they invite for interview?

10. Describe each of the following distributions.

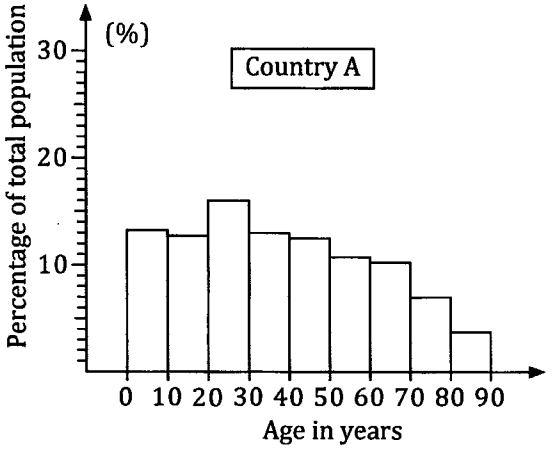
(a)

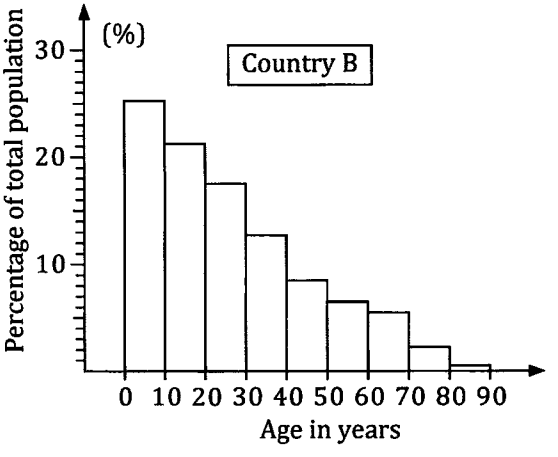


(b)



11. The graphs below show the age distribution of the human population of two countries, A and B.





(a) Which of the two countries has the greater population?

(b) If country A has a total population of approximately eighty two million people approximately how many of these are 70 or over?

(c) Produce a report describing the population distributions of each country, commenting on any similarities and differences between the distributions of the two countries and comment on possible implications for future government policy given these population distributions.